C3 June 2008
Model Solutions

① $y = 4e^{2x+1}$

a) If $y = 8$, $8 = 4e^{2x+1}$

$2 = e^{2x+1}$

$\ln 2 = 2x+1 \Rightarrow \dfrac{\ln 2 - 1}{2} = x$

b) $m_{tangent} = \left.\dfrac{dy}{dx}\right|_{x = \frac{\ln 2 - 1}{2}}$

$\dfrac{dy}{dx} = 8e^{2x+1}$, when $x = \dfrac{\ln 2 - 1}{2}$ $2x+1 = \ln 2$

$\Rightarrow \dfrac{dy}{dx} = 8e^{\ln 2} = 16.$

Tangent has equation $y - 8 = 16\left(x - \frac{\ln 2 - 1}{2}\right)$

$y = 16x - (8\ln 2 - 8) + 8$

$y = 16x + 16 - 8\ln 2$

$a = 16$, $b = 16 - 8\ln 2$

② a) $f(x) = 5\cos x + 12\sin x$

$f(x) = R\cos(x - \alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

$R\cos\alpha = 5$

$R\sin\alpha = 12$

$R^2 = 5^2 + 12^2 \Rightarrow R = 13$

$\tan\alpha = {}^{12}/_{5} \Rightarrow \alpha = 1.176$

$f(x) = 13\cos(x - 1.176)$

b) $5\cos x + 12\sin x = 6$

$0 \le x < 2\pi$

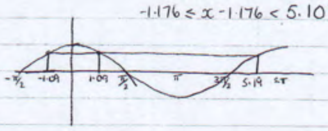
$-1.176 \le x - 1.176 < 5.107$

$13\cos(x - 1.176) = 6$

$\cos(x - 1.176) = {}^{6}/_{13}$

$x - 1.176 = \cos^{-1}({}^{6}/_{13}) = 1.09,$

-1.09



$\Rightarrow x = 0.086, 2.266$

c) i) Maximum value $= 13$

ii) Maximum of $\cos x$ is at $x = 0$ or $x = 2\pi$

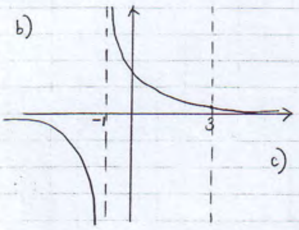
$\therefore$ maximum of $\cos(x - 1.176)$ is at $x = 1.176$

④ a) $f(x) = \dfrac{2(x-1)}{(x-3)(x+1)} - \dfrac{1}{(x-3)}$

$= \dfrac{2(x-1)}{(x-3)(x+1)} - \dfrac{(x+1)}{(x-3)(x+1)} = \dfrac{2(x-1)-(x+1)}{(x-3)(x+1)}$

$= \dfrac{\cancel{x-3}}{\cancel{(x-3)}(x+1)} = \dfrac{1}{x+1}$ QED

b)



When $x = 3$, $y = \frac{1}{3+1} = \frac{1}{4}$.

$\therefore$ Range is $0 < f(x) < \frac{1}{4}$

c) $y = \dfrac{1}{x+1}$ $x = \dfrac{1}{y+1} \Rightarrow xy + x = 1$

$xy = 1 - x$

$f^{-1}(x) = y = \dfrac{1-x}{x}$

Domain is $0 < x < \frac{1}{4}$

d) $g(x) = 2x^2 - 3$ $fg(x) = \dfrac{1}{(2x^2-3)+1} = \dfrac{1}{2x^2-2}$

$fg(x) = \frac{1}{8}$

$\Rightarrow \dfrac{1}{2x^2-2} = \dfrac{1}{8} \Rightarrow 2x^2 - 2 = 8 \Rightarrow 2x^2 = 10$

$x = \pm\sqrt{5}$

⑤ a) $\sin^2\theta + \cos^2\theta \equiv 1$

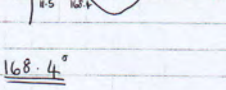
$(\div \sin^2\theta)$ $\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} \equiv \dfrac{1}{\sin^2\theta} \Rightarrow 1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$ QED

b) $2\cot^2\theta - 9\operatorname{cosec}\theta = 3$

$2(\operatorname{cosec}^2\theta - 1) - 9\operatorname{cosec}\theta = 3$

$2\operatorname{cosec}^2\theta - 9\operatorname{cosec}\theta - 5 = 0$

$(2\operatorname{cosec}\theta + 1)(\operatorname{cosec}\theta - 5) = 0$



$\Rightarrow \operatorname{cosec}\theta = -\frac{1}{2}$ or $\operatorname{cosec}\theta = 5$

$\sin\theta = -2$ ✗ $\quad \sin\theta = \frac{1}{5} \Rightarrow \theta = 11.5°, 168.4°$

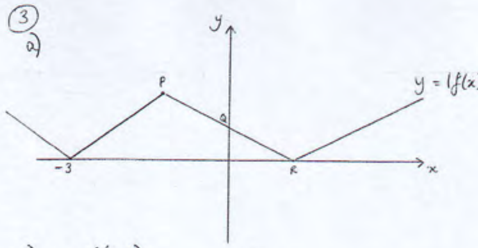
⑥ a) i) $y = e^{3x}(\sin x + 2\cos x)$ $\quad u = e^{3x}$ $\quad v = \sin x + 2\cos x$

$y' = e^{3x}(\cos x - 2\sin x) + 3e^{3x}(\sin x + 2\cos x)$ $\quad u' = 3e^{3x}$ $\quad v' = \cos x - 2\sin x$

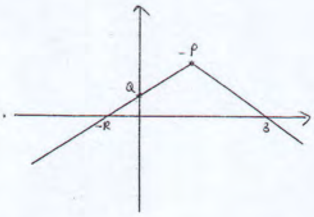
$= e^{3x}(\cos x - 2\sin x + 3\sin x + 6\cos x)$

$= e^{3x}(7\cos x + \sin x)$

③ a)



b) $y = f(-x)$



c) $f(x) = 2 - |x+1|$

At R, $y = 0$: $\quad 2 - |x+1| = 0$

$|x+1| = 2$

$\Rightarrow x = 1$ $\qquad R = (1, 0)$

At Q, $x = 0$: $\quad 2 - |0+1| = y$

$y = 2 - 1 = 1$ $\qquad Q = (0, 1)$

$f(x)$ takes $|x+1|$, which has minimum value at $(-1, 0)$ and reflects in x-axis before moving up 2, $\therefore$ P is the point $(-1, 2)$

d) $f(x) = \frac{1}{2}x$ Intersection is in 2 places:

Ⓐ $2 - (x+1) = \frac{1}{2}x$

$1 = \frac{3}{2}x \Rightarrow x = \frac{2}{3}$

Ⓑ $2 - -(x+1) = \frac{1}{2}x$

$2 + (x+1) = \frac{1}{2}x$

$\frac{1}{2}x = -3$

$x = -6$

⑥ a)

ii) $y = x^3\ln(5x+2)$ $\quad u = x^3$ $\quad v = \ln(5x+2)$

$u' = 3x^2$ $\quad v' = \dfrac{5}{5x+2}$

$y' = \dfrac{5x^3}{5x+2} + 3x^2(\ln(5x+2)) = x^2\left(\dfrac{5x}{5x+2} + 3\ln(5x+2)\right)$

b) $y = \dfrac{3x^2+6x-7}{(x+1)^2}$ $\quad u = 3x^2+6x-7$ $\quad v = (x+1)^2$ $\quad v^2 = (x+$

$u' = 6x + 6 = 6(x+1)$ $\quad v' = 2(x+1)$

$y' = \dfrac{6(x+1)^3 - 2(x+1)(3x^2+6x-7)}{(x+1)^4} = \dfrac{\cancel{(x+1)}[6(x+1)^2 - 2(3x^2+6x-7)]}{(x+1)^{4\,3}}$

$= \dfrac{6(x^2+2x+1) - 2(3x^2+6x-7)}{(x+1)^3}$

$= \dfrac{20}{(x+1)^3}$ QED

c) $\dfrac{dy}{dx} = 20(x+1)^{-3}$

$\dfrac{d^2y}{dx^2} = -60(x+1)^{-4} = \dfrac{-60}{(x+1)^4}$

If $\dfrac{-60}{(x+1)^4} = -\dfrac{15}{4} \Rightarrow (x+1)^4 = 16$

$x + 1 = \pm 2$

$x = 1$ or $x = -3$

⑦ $f(x) = 3x^3 - 2x - 6$

a) $f(1.4) = -0.568\ (<0)$

$f(1.45) = 0.245875\ (>0)$

Change of sign $\Rightarrow$ root between 1.4 and 1.45.

b) $3x^3 - 2x - 6 = 0$ $\qquad 3x^3 = 2x + 6$

$x^3 = \frac{2x}{3} + 2$

$x^2 = \frac{2}{3} + \frac{2}{x} \Rightarrow x = \sqrt{\frac{2}{x} + \frac{2}{3}}$ QED.

c) $x_0 = 1.43$

$x_1 = 1.4371$

$x_2 = 1.4347$

$x_3 = 1.4355$

d) $1.4345 < 1.435 < 1.4355$

$f(1.4345) = -0.013\ (<0)$

$f(1.4355) = 0.0032\ (>0)$

Change of sign $\Rightarrow$ root in interval.